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NRL Report 5917

A Numerical Method for the Transient Response of Nonlinear Systems

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June 21, 1963

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Washington, D.C.

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SYMBOLS

c_i	viscous damping constant between m_i and m_{i-1}
F_i	forcing function on m_i
f_{10}	nonlinear spring force between mass and foundation; $kx + \beta x^3$
f_{21}	nonlinear spring force between m_2 and m_1; $k_2x_2 + \beta_2 x_2^3$
$\bar{f}_n$	mean value of the modified nonlinear spring force for $t_n < t < t_{n+1}$
h	finite time increment; Δt
k_i	spring constant between m_i and m_{i-1}
m_i	*ith* mass
p	$\omega\sqrt{1-\alpha^2}$
r	$\sqrt{1-\alpha^2}$
S_n	$F_{n+1} - F_n$
S^2_{n-1}	$S_n - S_{n-1} = F_{n+1} - 2F_n + F_{n-1}$
$\mathbf{S}_n$	$\dot{Z}_{n+1} - \dot{Z}_n$
$\mathbf{S}^2_{n-1}$	$\mathbf{S}_n - \mathbf{S}_{n-1} = \dot{Z}_{n+1} - 2\dot{Z}_n + \dot{Z}_{n-1}$
t	independent time variable
u	scaled velocity $\dot{x}/\omega$ for one-degree-of-freedom system
u_n	scaled velocity at $t = t_n$ for one-degree-of-freedom system
u_i	scaled velocity $\dot{x}_i/\omega_i$
$(u_i)_n$	scaled velocity at $t = t_n$
v	scaled velocity $\dot{y}/\omega$ for one-degree-of-freedom system
v_n	scaled velocity at $t = t_n$ for one-degree-of-freedom-system
v_i	scaled velocity $\dot{y}_i/\omega_i$
$(v_i)_n$	scaled velocity at $t = t_n$
x	relative displacement between mass and foundation for one-degree-of-freedom system
x_n	relative displacement between mass and foundation at $t = t_n$ for one-degree-of-freedom system
x_i	relative displacement between m_i and m_{i-1}
$(x_i)_n$	relative displacement between m_i and m_{i-1} at $t = t_n$
y	absolute displacement of mass in one-degree-of-freedom system
y_n	absolute displacement of mass at $t = t_n$ for one-degree-of-freedom system
y_i	absolute displacement of m_i
$(y_i)_n$	absolute displacement of m_i at $t = t_n$
z	absolute foundation displacement
α_i	$c_i/2m_i\omega_i$
β_i	coefficient of nonlinear term of spring
γ_i^2	β_i/m_i
δ_i	delta term for the *ith* equation of motion
θ	ωh
ω_i^2	k_i/m_i

A Numerical Method for the Transient Response of Nonlinear Systems

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Numerical integration equations are derived for determining the response of nonlinear systems subjected to transient loads. The numerical method consists of approximating the nonlinear variables and the forcing functions in the differential equations over a short interval of time by their mean value, by a straight line, or by a parabola. This allows for Duhamel integral type solutions for the nonlinear terms. A step by step solution follows which uses an iteration method during each increment of the solution. The sufficient condition for the convergence of the iteration method is presented for the case of N numerical equations. A scaling law is presented which eliminates linear damping from the equation of motion by a prescribed transformation. Example problems of a one-degree-of-freedom system and a two-degree-of-freedom system are solved by the numerical integration equations and the solutions are compared with response curves obtained from analog computers at NRL.

INTRODUCTION

This report deals primarily with approximate numerical solutions of a single or a set of nonautonomous second-order ordinary nonlinear differential equations. While the class of problems under consideration lie in the field of structural dynamics, the proposed solutions are applicable to many other physics and engineering fields. The mathematical tools of ordinary nonlinear differential equations are particularly useful when dealing with autonomous solutions or with approximate steady-state solutions of these problems. However, they require considerable ingenuity and insight to apply and are not suited for the study of transient behavior. An attempt is made in this report to present an easily understood, yet powerful and precise, technique which will allow most engineers to cope with the transient response of nonlinear systems.

Two previous NRL Reports (1,2) have dealt with this problem, and this report pursues the same general approach. Most approximate numerical techniques fail to attack directly the nonlinear differential equations in their solutions. Rather, they introduce Maclaurin or Taylor series expansions of the functions as in the method of Picard (3). The other general approach is to replace differential equations by equations of finite differences and to use these equations as an approximation to the differential equations. These are good general purpose techniques. However, they tend to be routine techniques which remove the analyst from a clear understanding of the manner in which the differential equations were solved.

The numerical method presented uses only those mathematical tools which are familiar to most engineering graduates and are applied directly to the class of differential equations under study. It should not be construed that this is a crude technique and that the solutions will be greatly in error or will have inherent instabilities of large magnitude. The examples in this report show the opposite to be true.

To those readers who are already familiar with Refs. 1 and 2, this report is a direct application of the principles explained therein. For those persons who have not read them, however, it is noted that this report is completely self-contained and these references are not required reading.

BACKGROUND THEORY

The Linear Problem

It will be beneficial to review a numerical integration method (1) which is used to solve linear single-degree-of-freedom problems before proceeding to the nonlinear ones. Consider the undamped linear oscillator shown in Fig. 1 subject

NRL Problem R05-24B, Project WW 041. This is an interim report on one phase of the problem; work is continuing.

Manuscript submitted December 11, 1962.

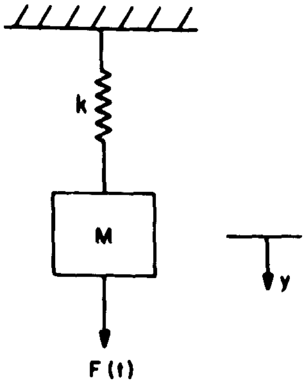


Fig. 1 - Linear oscillator

to an applied force $F(t)$. Let y be the displacement of the oscillator so that the differential equation of motion is

$$m\ddot{y} + ky = F(t) \tag{1}$$

where the dots denote differentiation with respect to time. If $\omega^2 = k/m$, Eq. (1) becomes

$$\ddot{y} + \omega^2 y = \frac{F(t)}{m}. \tag{2}$$

The general solution of this equation is

$$y = y_c + y_p \tag{3}$$

where y_c is the complementary solution and y_p is the particular solution. This property of linear equations will be shown to have some value in the approximate solution of nonlinear equations.

For the case under study the complementary solution is well known and the particular solu- is a Duhamel integral (4). If $v = \dot{y}/\omega$, the general solution of Eq. (2) is

$$y = y_0 \cos \omega t + v_0 \sin \omega t + \frac{1}{m\omega}\int_0^t F(T) \sin \omega(t - T)\, dT \tag{4a}$$

and its scaled derivative v is

$$v = -y_0 \sin \omega t + v_0 \cos \omega t + \frac{1}{m\omega}\int_0^t F(T) \cos \omega(t - T)\, dT \tag{4b}$$

where y_0 and v_0 are the initial values at $t = t_0 = 0$.

Usually the integrals of Eq. (4) cannot be evaluated for an arbitrary curve of F. If F is divided into equal segments of time,* and represented in some approximate manner for each increment, a step by step approximate solution follows. It is noted that Eq. (4) is true for all times during the response of the oscillator. For example, suppose $y = y_1$ and $v = v_1$ at $t = t_1$. Now the time can be redefined arbitrarily to start at zero for the next increment with y_1 and v_1 being the initial conditions. The Duhamel integrals are solved for this next step, y_2 and v_2 are found, and a repetition of the process defines the next pair of points. The process is self-perpetuating.

The problem in this direct attack upon the differential equation of motion has resolved itself into the solution of these integrals for a short time increment. Since the forcing function may be known only as a graphical function, as a discontinuous function, or as a complicated analytic function, some methods of describing it over the immediate range of integration is now discussed.

Approximate Methods of Representing Functions

Three methods are presented for the approximate representation of a function over finite increments of time $\Delta t = h$. Suppose a portion of an arbitrary function $F(t)$ is divided into equal segments of time.* Figure 2 shows the rectangular step representation consisting of horizontal lines drawn through the mean value of the function over each increment h. Appendix A reviews two common procedures for obtaining graphically the mean value of a function for a given increment. The equation of the function during each increment is

$$F(t) = \overline{F}_n = \text{constant}, \quad t_n < t < t_{n+1} \tag{5}$$

where $\overline{F}_n$ is the mean value of F during the increment.

The second method represents the curve by a straight line through the end points across each increment as shown in Fig. 3. The equation of the function from t_n to t_{n+1} is

$$F(t) = F_n + S_n \frac{t}{h} \tag{6}$$

where $S_n = F_{n+1} - F_n$, and time begins at t_n.

*Dividing F into equal segments of time is an unnecessary but convenient restriction since it makes calculations easier and computer programming less cumbersome.

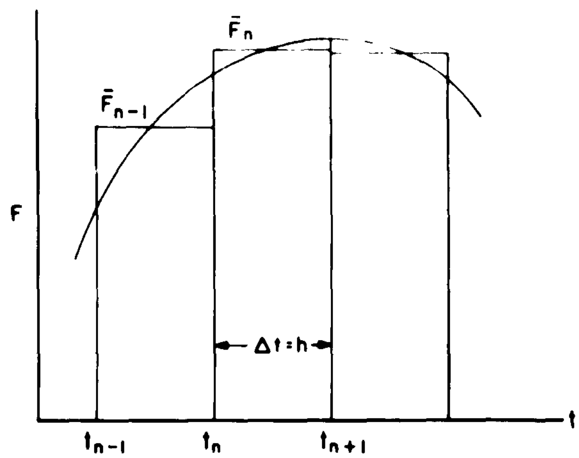


Fig. 2 - Representation of a function by rectangular steps

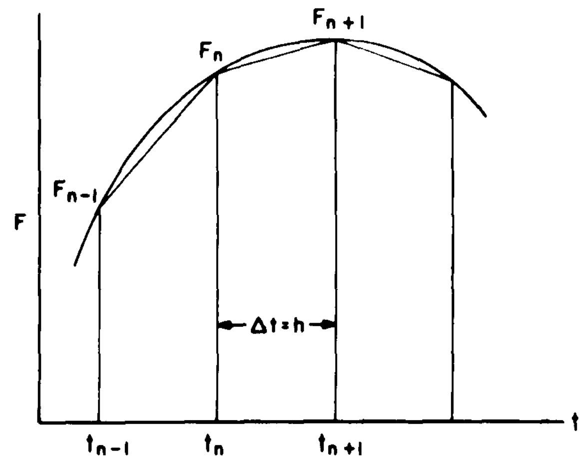


Fig. 3 - Representation of a function by straight lines

The third method of approximating a function across an increment is to pass a parabola through three successive points. For example, Fig. 3 shows three successive points on the curve, namely, F_{n-1}, F_n, and F_{n+1}. The equation of the parabola from t_n to t_{n+1} passing through these three points on the curve is

$$F(t) = F_n + S_n \frac{t}{h} + \frac{S_n^2}{2}\left(\frac{t^2}{h^2} - \frac{t}{h}\right) \qquad (7a)$$

or

$$F(t) = F_n + S_n \frac{t}{h} + \frac{S_{n-1}^2}{2}\left(\frac{t^2}{h^2} - \frac{t}{h}\right) \qquad (7b)$$

where

$$S_n^2 = S_{n+1} - S_n = F_{n+2} - 2F_{n+1} + F_n \qquad (8)$$

$$S_{n-1}^2 = S_n - S_{n-1} = F_{n+1} - 2F_n + F_{n-1} \qquad (9)$$

and time again begins at t_n. Note that S^2 is *not* the square of S. Equation (7a) is used to represent the curve during the first increment of time, that is, for $n = 0$. Equation (7b) is the expression used for the remaining segments of the curve. Equation (7a) is derived specially to avoid the nonexistent term in S^2_{-1} (that is, F_{-1}) if Eq. (7b) were used for the first increment.

For the case where two successive increments are not equal, say h between F_{n-1} and F_n and h' between F_n and F_{n+1}, Eq. (7b) is adjusted to read

$$F(t) = F_n + \left\{\frac{-(h')^2F_{n-1} + [(h')^2 - h^2]F_n + h^2F_{n+1}}{hh'(h+h')}\right\}t + \left[\frac{h'F_{n-1} - (h+h')F_n + hF_{n+1}}{hh'(h+h')}\right]t^2. \qquad (10)$$

Solution Equations for the Undamped Linear Oscillator

If the parabolic representation of a function given by Eq. (7b) is substituted for F in Eqs. (4), and the integrations performed and evaluated at $t = h$, there results

$$y_{n+1} = y_n \cos\theta + v_n \sin\theta + \frac{F_n}{k}(1 - \cos\theta) + \frac{S_n}{k}\left(1 - \frac{\sin\theta}{\theta}\right) + \frac{S_{n-1}^2}{2k}\left[\frac{\sin\theta}{\theta} - \frac{2(1-\cos\theta)}{\theta^2}\right] \qquad (11a)$$

$$v_{n+1} = -y_n \sin\theta + v_n \cos\theta + \frac{F_n}{k}\sin\theta + \frac{S_n}{k}\frac{(1-\cos\theta)}{\theta} + \frac{S_{n-1}^2}{2k}\left(\frac{1+\cos\theta}{\theta} + \frac{2\sin\theta}{\theta^2}\right) \qquad (11b)$$

where $\theta = \omega h$. It is noted that for constant increments h, the trigonometric coefficients are calculated just once for the entire solution.

Equations (11) are used for the straight line representation of F by setting S^2 equal to zero, and for the rectangular step representation by setting both S and S^2 equal to zero. The equations for the latter case are

$$y_{n+1} = y_n \cos\theta + v_n \sin\theta + \frac{\bar{F}_n}{k}(1-\cos\theta) \tag{12a}$$

$$v_{n+1} = -y_n \sin\theta + v_n \cos\theta + \frac{\bar{F}_n}{k}\sin\theta. \tag{12b}$$

Recall that $\bar{F}_n$ is the mean value of the function during each increment. For functions such as F, which are explicit functions of time, it is recommended to find the average value of F during the time increment instead of employing the graphical techniques outlined in Appendix A. The average value of the function represented by a straight line is

$$\bar{F}_n = \frac{F_n + F_{n+1}}{2} \tag{13}$$

while the average value for a parabolic representation is

$$\bar{F}_n = \frac{1}{12}(-F_{n-1} + 8F_n + 5F_{n+1}). \tag{14}$$

Either Eq. (13) or (14) is then substituted for $\bar{F}_n$ in Eqs. (12) and the numerical integral equations solved in a step by step fashion.

Solution Equations for the Viscously Damped Linear Oscillator

The method of solution for the undamped case is also applicable to the viscously damped system. The equation of motion of an oscillator with linear damping is

$$m\ddot{y} + c\dot{y} + ky = F(t). \tag{15}$$

If $\alpha = c/2m\omega$, Eq. (15) becomes

$$\ddot{y} + 2\alpha\omega\dot{y} + \omega^2 y = \frac{F(t)}{m} \tag{16}$$

where $-\infty < \alpha < \infty$, and $\omega > 0$.

The form of the general solution for Eq. (16) is given by Eq. (3) which, for this case, consists of a combination of exponential, trigonometric, and hyperbolic functions, and the solutions may be found in Appendix B. The forcing function is approximated as previously discussed and the solution carried on precisely as in the undamped case. For example, with $\alpha = 1$ and the average or mean value of the force used, the equations are

$$y_{n+1} = y_n(1+\theta)e^{-\theta} + v_n\theta e^{-\theta} + \frac{\bar{F}_n}{k}[1-(1+\theta)e^{-\theta}]$$

$$v_{n+1} = -y_n\theta e^{-\theta} + v_n(1-\theta)e^{-\theta} + \frac{\bar{F}_n}{k}\theta e^{-\theta}.$$

Comments

The solution of the linear problem leads to several interesting observations.

1. A direct attack on the differential equation is made.

2. The coefficients of the variable terms in the numerical equations become constants throughout the solution when equal time intervals are employed.

3. The solution makes use of the natural expansion functions for the differential equations. That is, they are of the form of trigonometric, hyperbolic, and exponential functions as they would be in an analytical case.

NONLINEAR PROBLEM

Numerical Solution

Suppose an ordinary second-order differential equation is reducible to the form

$$\ddot{y} + H(y, \dot{y}, t) = 0 \tag{17}$$

where $H(y, \dot{y}, t)$ may be a very complicated function. Since the function H contains a forcing function F, Eq. (17) becomes

$$\ddot{y} + G = \frac{F}{m} \tag{18}$$

where

$$H = G - \frac{F}{m}.$$

Let Eq. (18) be rewritten

$$\ddot{y} + 2\alpha\omega\dot{y} + \omega^2 y = \frac{F}{m} - [G - 2\alpha\omega\dot{y} - \omega^2 y]$$

and let

$$\omega^2\delta = G - 2\alpha\omega\dot{y} - \omega^2 y$$

so that

$$\ddot{y} + 2\alpha\omega\dot{y} + \omega^2 y = \frac{F}{m} - \omega^2\delta. \qquad (19a)$$

If δ were zero for all time, the equation would be linear, and the solution has already been presented. For convenience let the damping term α be zero in Eq. (19a). Then,

$$\ddot{y} + \omega^2 y = \frac{F}{m} - \omega^2\delta. \qquad (19b)$$

The solution of this equation is similar to Eqs. (4); that is,

$$y = y_0 \cos\theta + v_0 \sin\theta + \frac{1}{m\omega}\int_0^t F(T) \sin\omega(t-T)dT - \omega\int_0^t \delta(T)\sin\omega(t-T)dT \qquad (20a)$$

$$v = -y_0 \sin\theta + v_0 \cos\theta + \frac{1}{m\omega}\int_0^t F(T) \cos\omega(t-T)dT - \omega\int_0^t \delta(T)\cos\omega(t-T)dT. \qquad (20b)$$

The solution of the linear problem for a given arbitrary curve of F requires that F be partitioned into finite increments of time and be approximated over each increment by one of several representations. The same approach, using the same increment h, is proposed to handle the integrals in Eqs. (20) containing the nonlinear terms in δ.

For example, each term in δ might be approximated by the parabolic representation so that numerical integration equations for y and v are obtained. Consider a simple oscillator with a cubic hardening spring. The equation of motion is

$$m\ddot{y} + ky + \beta y^3 = F(t)$$

or

$$\ddot{y} + \omega^2 y = \frac{F(t)}{m} - \omega^2\delta$$

where

$$\delta = \frac{\beta}{k} y^3.$$

Use Eq. (7b) for the cubic term, obtaining

$$\delta = \frac{\beta}{k}\left[y_n + (y_{n+1} - y_n)\frac{t}{h} + \frac{(y_{n+1} - 2y_n + y_{n-1})}{2}\left(\frac{t^2}{h^2} - \frac{t}{h}\right)\right]^3. \qquad (21)$$

Expand Eq. (21), substitute it into Eqs. (20) for δ, and integrate each term over the increment. If the forcing function were also approximated by the parabolic representation, the resulting numerical integration equations are of the form of Eqs. (11) with the additional terms y_{n-1}, y_n, and y_{n+1}; that is,

$$y_{n+1} = g_1(F_{n-1}, F_n, F_{n+1}, y_{n-1}, y_n, y_{n+1}, v_n, h) \qquad (22a)$$

$$v_{n+1} = g_2(F_{n-1}, F_n, F_{n+1}, y_{n-1}, y_n, y_{n+1}, v_n, h). \qquad (22b)$$

Everything is known on the right-hand side of Eqs. (22) except y_{n+1}. As a first trial this value may be assumed equal to y_n. Substitute this into the right-hand side of Eq. (22a) to find y_{n+1}. Use this value for y_{n+1} in the right-hand side of Eq. (22a) to find a second value of y_{n+1}. Repeat this iteration process until the succeeding values of y_{n+1} converge. Use the final value of y_{n+1} in Eq. (22b) to find v_{n+1}.

This method of solution could also be used if instead of the parabolic representation the straight line representation approximated the nonlinear term. However, in either case a great deal of time and effort is required if the analyst uses a desk calculator. Of course, the numerical equations could be programmed for an electronic computer. Everything which follows from this point is directed toward desk calculator computations.

The recommended method of solution with a desk calculator is to use the mean value or average value of the variables in the δ term during a given finite increment. If this assumption is also extended to the forcing function, Eqs. (20) are integrated to give

$$y_{n+1} = y_n \cos\theta + v_n \sin\theta + \frac{\overline{F}_n}{k}(1-\cos\theta) - \delta_n(1-\cos\theta) \quad (23a)$$

$$v_{n+1} = -y_n \sin\theta + v_n \cos\theta + \frac{\overline{F}_n}{k}\sin\theta - \delta_n \sin\theta. \quad (23b)$$

For the oscillator with the cubic hardening spring, the straight line averaging method yields

$$\delta_n = \frac{\gamma^2}{8}(y_n + y_{n+1})^3 \quad (24)$$

while the parabolic averaging method yields

$$\delta_n = \frac{\gamma^2}{1728}(-y_{n-1} + 8y_n + 5y_{n+1})^3. \quad (25)$$

Once again y_{n+1} is the unknown on the right-hand side of Eqs. (23) and the iteration process is used for each step of the solution as previously described.

The First Increment

Special treatment is necessary for the application of the numerical integration equations at the start of the solution. At the beginning of a problem the initial conditions are always known so that y_0 and v_0 are established. Having selected a time increment h, the first step of the solution depends upon the type of averaging method to be used for the nonlinear terms. In the case of the straight line averaging method, the first trial value for y_1 might be assumed; y_1 might be set equal to y_0, or a Maclaurin series might be used to approximate y_1. In any event, the first trial value of y_1 is substituted into the right-hand side of the numerical integration equation to find a new value of y_1. The iteration procedure follows as previously mentioned. Of course, if δ contains the scaled velocity, the same approach is used to find v_1.

In the case of the parabolic averaging method, Eq. (25) shows the rear point y_{n-1}, which must be known before iterating to find y_{n+1}. At the start of a problem where $n = 0$, y_{-1} does not exist. It is suggested that the straight line average be used for the first increment to establish y_1. If greater accuracy is desired, use the linear averaging method for half an increment, that is, for $h/2$. Having established $y_{1/2}$ in this manner, the parabolic averaging method is now used for another half increment to find y_1. The full increment might then be used from this point throughout the remainder of the solution.

Graphical Form of Nonlinear Components

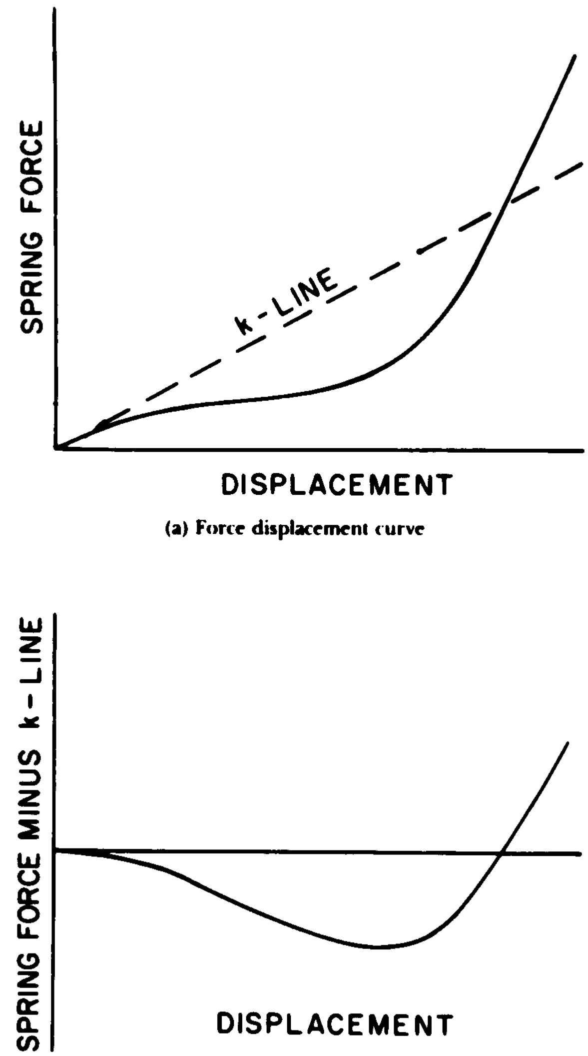
Quite frequently the nonlinear characteristic of a material in a system is determined from laboratory experiments and is plotted as force versus displacement or velocity. It is sometimes possible to find an analytical expression for such a curve. In the event this is not readily attainable, a graphical technique for finding the mean value of a function over an increment is used (see Appendix A).

For purposes of illustration consider Fig. 4a, which shows the spring force for positive displacements only. A line tangent to the curve at the origin is drawn and is labeled the k-line. Figure 4b represents the spring force minus the k-line as a function of displacement. This is the curve from which the mean value of f is determined. Suppose an oscillator contains such a spring. Equations (23) become

$$y_{n+1} = y_n \cos\theta + v_n \sin\theta + \frac{\overline{F}_n}{k}(1-\cos\theta) - \frac{\overline{f}_n}{k}(1-\cos\theta) \quad (26a)$$

$$v_{n+1} = -y_n \sin\theta + v_n \cos\theta + \frac{\overline{F}_n}{k}\sin\theta - \frac{\overline{f}_n}{k}\sin\theta \quad (26b)$$

where $\overline{F}_n$ is the mean value of the forcing function and $\overline{f}_n$ is the mean value of the curve shown in Fig. 4b. At step n, y_n and v_n are known, $\overline{F}_n$ is known from the input curve, $\overline{f}_n$ depends upon y_n and the



(a) Force displacement curve

(b) Adjusted force displacement curve

Fig. 4 - Spring force of a nonlinear spring. The k-line is tangent to the curve at the origin.

unknown y_{n+1}. As a first trial for finding y_{n+1}, find f for y_n and use this for f_n. Substitute into Eq. (26a) to find a first trial of y_{n+1}. Now find f_n between y_n and y_{n+1} from Fig. 4b and substitute this into Eq. (26a) to obtain a new value of y_{n+1}. Repeat the process until succeeding values of y_{n+1} converge.

For certain curve shapes it might be advantageous to draw two or more k-lines for certain portions of a given curve. These k-lines would provide a solution which follows more closely a piecewise linear solution of the system by reducing the magnitude of the adjusted forces f. This means that a corresponding number of differential equations must be written for each region of the curve where a k-line is drawn. Proper initial conditions and ω's must be determined for each numerical integration equation. An example might be a material whose force-displacement curve follows closely an ideal elastic-plastic relationship.

Forcing Functions

Foundation motion of structures is an important type of forcing function in the field of structural dynamics. Such motion may be described as foundation acceleration, velocity, or displacement. In the case of foundation acceleration the differential equation of motion for a nonlinear oscillator with a cubic hardening spring is

$$\ddot{x} + \omega^2 x = -\ddot{z} - \gamma^2 x^3 \tag{27}$$

where x is the relative displacement between the mass and the foundation and $\ddot{z}$ is the foundation acceleration. This equation is similar to Eq. (2), with x replacing y and $-\ddot{z}$ replacing F/m. The parabolic averaging method is recommended for systems with a known curve for $\ddot{z}$. This average for $\ddot{z}$ is the same as Eq. (14) provided the F terms are replaced by $\ddot{z}$ terms. Equations (11) may be also used provided the following changes are made:

$$F_n = -m\ddot{z}_n,\ \frac{F_n}{k} = -\frac{\ddot{z}_n}{\omega^2},\ \frac{S_n}{k} = -\frac{S_n}{\omega^2},$$

$$\frac{S^2_{n-1}}{k} = -\frac{S^2_{n-1}}{\omega^2},\ y_n = x_n,\ v_n = u_n.$$

When the foundation velocity is the prescribed input, an interesting relationship is found for the parabolic average of $\ddot{z}$ to be used in the numerical solution of Eq. (27). Consider the parabolic representation for foundation velocity and differentiate to find foundation acceleration:

$$\dot{z} = \dot{z}_n + S_n \frac{t}{h} + \frac{S^2_{n-1}}{2}\left(\frac{t^2}{h^2} - \frac{t}{h}\right)$$

$$\ddot{z} = \frac{S_n}{h} + \frac{S^2_{n-1}}{h^2}\,t - \frac{S^2_{n-1}}{2h}.$$

The average acceleration during the increment is

$$\overline{\ddot{z}} = \frac{1}{h}\int_0^h \ddot{z}\; dt = \frac{S_n}{h} = \frac{\dot{z}_{n+1} - \dot{z}_n}{h}.$$

This is the approximate representation for the foundation acceleration according to the theorem of the mean of differential calculus.

If the foundation displacement is the prescribed input, the differential equation of motion for the nonlinear oscillator should be of the form

$$\ddot{y}_1 + \omega^2 y_1 = \omega^2 z - \gamma^2 (y_1 - z)^3 \qquad (28)$$

where y_1 is the absolute displacement of the mass. The parabolic average representation for z is found for each increment from the given curve.

Step Changes in Input

The numerical integration equations are particularly applicable to problems in which step changes occur in the input of the system. The only restriction in the solution is to require that a given increment terminate where the step change occurs. Let the curve shown in Fig. 5 be a foundation velocity of a nonlinear oscillator, and suppose the increment h is selected for the solution. After the fifth step of the solution, a new increment h' is used for the sixth increment in order to arrive at the time when the step change in the foundation velocity occurs. The size increment h is then used throughout the remainder of the solution.

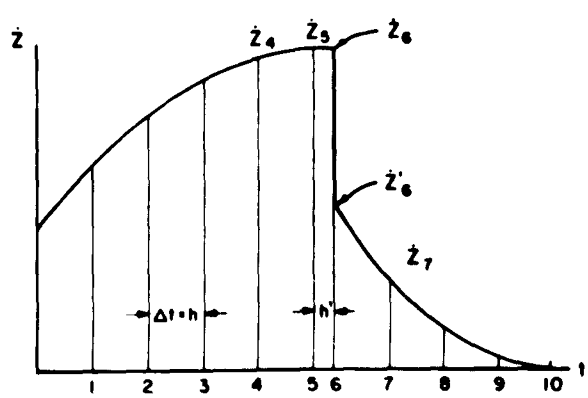


Fig. 5 - Step change in foundation velocity

For the sixth increment of the numerical integral solution, a new increment $\theta' = \omega h'$ must replace $\theta = \omega h$ in the solution equations. Equation (10) must be used if the input is being approximated by the parabolic representation. In so doing, the solution is found at step 6 for x_6 and u_6, where x_6 is the relative displacement between the mass of the oscillator and the foundation and u_6 is the corresponding scaled relative velocity. At this point the step change in foundation velocity (scaled) must be added to u_6 to give u_6'. With x_6 and u_6' as the initial conditions for step 7 of the solution, and using θ in the solution equations, the input curve should be represented by the straight line method between points 6 and 7 instead of the parabolic method due to the discontinuity at point 6. This completes the solution to step 7. The parabolic representation may be used from this point to the end of the input curve.

CONVERGENCE OF THE ITERATION METHOD

One may wonder if an assumed variable could not produce a diverging solution when substituted into the δ term on the right-hand side of the numerical equations for a given increment. Scarborough (3b) has discussed the sufficient conditions for convergence of one and two numerical equations when the iteration process is applied. An extension of the procedure follows for a set of N numerical equations.

Consider the set of equations

$$\epsilon_i = B_i[\epsilon_1, \epsilon_2, \ldots, \epsilon_N],\; i = 1, 2, \ldots, N \quad (29)$$

where B_i is a set of known functions in terms of the ϵ_i's. The set of equations is satisfied by the exact values of the set of roots ϵ_i. As a first approximation to find a set of roots, try $\epsilon_i^{(0)}$. Hence, Eq. (29) gives

$$\epsilon_i^{(1)} = B_i[\epsilon_1^{(0)}, \epsilon_2^{(0)}, \ldots, \epsilon_N^{(0)}]. \qquad (30)$$

Substract Eq. (30) from (29) and apply the theorem of the mean value for a function of N variables. This gives

$$\epsilon_i - \epsilon_i^{(1)} = \sum_{j=1}^{N} (\epsilon_j - \epsilon_j^{(0)})\, B_{ij} \qquad (31)$$

where

$$B_{i,j} = \frac{\partial B_i[\epsilon_1^{(0)} + \phi(\epsilon_1 - \epsilon_1^{(0)}), \ldots, \epsilon_N^{(0)} + \phi(\epsilon_N - \epsilon_N^{(0)})]}{\partial \epsilon_j},$$

$$0 \leq \phi \leq 1.$$

Add each equation of the set of equations as expressed in Eq. (31) and consider only the absolute values. Thus,

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(1)}| \leq \sum_{i=1}^{N} \sum_{j=1}^{N} |\epsilon_j - \epsilon_j^{(0)}||B_{i,j}|. \qquad (32)$$

Let the maximum value of the terms $\{|B_{1,1}| + \ldots + |B_{N,1}|\}, \ldots, \{|B_{1,N}| + \ldots + |B_{N,N}|\}$ be a proper fraction ψ for all points in the region $(\epsilon_i^{(0)}, \epsilon_i)$. Then Eq. (32) becomes

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(1)}| \leq \psi \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(0)}|. \qquad (33)$$

This relation holds for the first approximation. For succeeding approximations, similar relations are obtained. That is,

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(2)}| \leq \psi \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(1)}|$$

$$\vdots \qquad \vdots$$

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(b)}| \leq \psi \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(b-1)}|. \qquad (34)$$

Multiply together all these inequalities as expressed in Eqs. (33) and (34) and divide through by the common factors

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(1)}|, \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(2)}|, \ldots, \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(b-1)}|$$

so that,

$$\sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(b)}| \leq \psi^b \sum_{i=1}^{N} |\epsilon_i - \epsilon_i^{(0)}|. \qquad (35)$$

Since ψ is a proper function, the right-hand member of this inequality may be taken as small as desired by repeating the iteration process a sufficient number of times. This means that the errors $|\epsilon_i - \epsilon_i^{(b)}|$ can be made as small as desired. Therefore, the iteration process converges when the N conditions

$$|B_{1,1}| + |B_{2,1}| + \ldots + |B_{N,1}| < 1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$|B_{1,N}| + |B_{2,N}| + \ldots + |B_{N,N}| < 1 \qquad (36)$$

are satisfied for all points in the neighborhood of $\epsilon_i^{(0)}$.

Consider Eqs. (23), which pertain to an increment of time h for a single-degree-of-freedom system. All terms on the right-hand side of the equations are constants except δ_n. Suppose δ_n is a function of displacement y and scaled velocity u. Equations (23) are of the form of Eq. (29) where

$$\epsilon_1 = y_{n+1}, \qquad \epsilon_2 = u_{n+1}$$

$$B_1 = y_n \cos\theta + v_n \sin\theta + \frac{\overline{F}_n}{k}(1 - \cos\theta) - \delta_n(1 - \cos\theta)$$

$$B_2 = -y_n \sin\theta + v_n \cos\theta + \frac{\overline{F}_n}{k}\sin\theta - \delta_n \sin\theta.$$

According to Eq. (36) the iteration process on the δ_n's converge in the nth step of the solution provided that

$$\left|(1 - \cos\theta)\frac{\partial \delta_n}{\partial y_{n+1}}\right| + \left|\sin\theta \frac{\partial \delta_n}{\partial y_{n+1}}\right| < 1$$

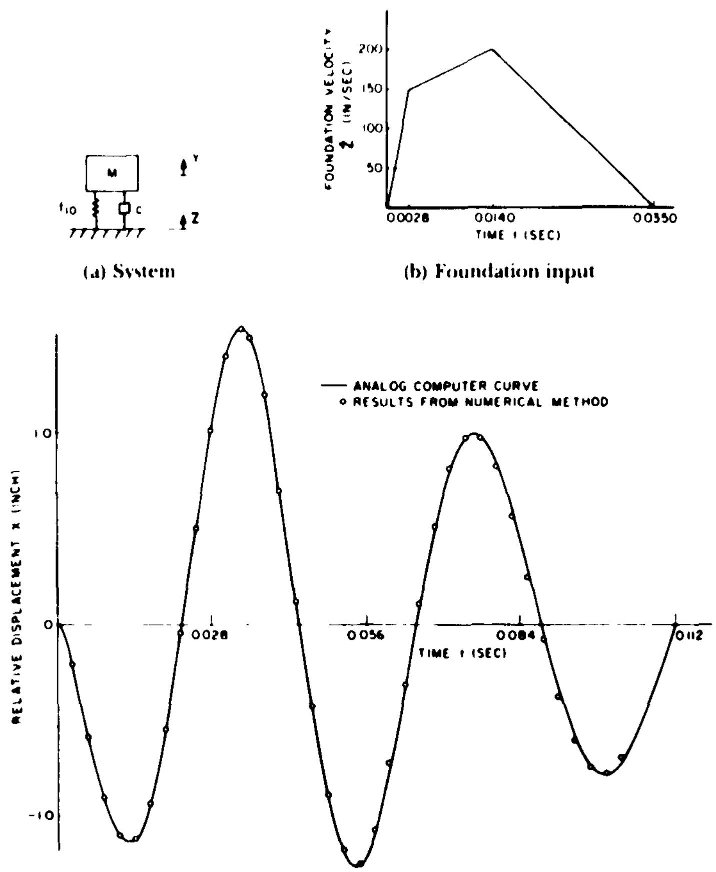
$$\left|(1 - \cos\theta)\frac{\partial \delta_n}{\partial v_{n+1}}\right| + \left|\sin\theta \frac{\partial \delta_n}{\partial v_{n+1}}\right| < 1$$

in the neighborhood of $[y_{n+1}^{(0)}, v_{n+1}^{(0)}]$.

A SCALING LAW

Consider the equation of motion for a single-degree-of-freedom system:

$$\ddot{y} + 2\alpha\omega\dot{y} + \omega^2 y = -\omega^2\delta \qquad (37)$$



(a) System

(b) Foundation input

(c) Response x vs t obtained from the numerical method and from an analog computer

Fig. 6 - Input and numerical solution of a nonlinear single-degree-of-freedom system with viscous damping

where $\delta = \delta(y, \dot{y}, t)$. Substitute the transformation

$$y = A e^{\alpha\omega t} \tag{38}$$

into Eq. (37). This gives

$$\ddot{A} + p^2 A = -\omega^2 \Delta e^{\alpha\omega t} \tag{39}$$

where $p^2 = \omega^2(1 - \alpha^2)$ and $\Delta = \Delta(A, \dot{A}, t)$.

Equation (39) can be solved graphically (2) or numerically and, using the transformation of Eq. (38), the response of the original system is found. For the case where the mean or average value of the terms in the Δ expression are used, the numerical integration equations for Eq. (39) are

$$A_{n+1} = A_n \cos ph + \frac{\dot{A}_n}{p} \sin ph - \overline{\Delta}_n e^{\alpha\omega(t_n + t_{n+1})/2} (1 - \cos ph) \tag{40a}$$

$$\frac{\dot{A}_{n+1}}{p} = -A_n \sin ph + \frac{\dot{A}_n}{p} \cos ph - \overline{\Delta}_n e^{\alpha\omega(t_n + t_{n+1})/2} \sin ph . \tag{40b}$$

EXAMPLES

Example 1

Figure 6a shows the single-degree-of-freedom system subjected to a transient foundation velocity shown in Fig. 6b. The equation of motion is

$$\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = -\omega^2\delta, \quad \alpha < 1$$

where $\delta = \beta x^3/k + \ddot{z}/\omega^2$. Since a desk calculator is used to find the response x of the system, the mean value of the terms in the δ expression is used. For this case $\alpha < 1$, so that the equations from Appendix B for Case II apply. By replacing F/m by $-\omega^2\delta$ in Eq. (B1), the numerical integration equations are

$$x_{n+1} = x_n e^{-\alpha\theta}\left(\cos ph + \frac{\alpha}{r}\sin ph\right) + u_n \frac{e^{-\alpha\theta}}{r}\sin ph - \delta_n\left[1 - e^{-\alpha\theta}\left(\cos ph + \frac{\alpha}{r}\sin ph\right)\right] \quad (41a)$$

$$u_{n+1} = -x_n \frac{e^{-\alpha\theta}}{r}\sin ph + u_n e^{-\alpha\theta}\left(\cos ph - \frac{\alpha}{r}\sin ph\right) - \delta_n \frac{e^{-\alpha\theta}}{r}\sin ph. \quad (41b)$$

The values of the parameters are

$$m = 0.10 \text{ lb-sec}^2/\text{in.} \qquad c = 2.4 \text{ lb-sec/in.}$$

$$k = 1440 \text{ lb/in.} \qquad \beta = 720 \text{ lb/in.}^3$$

A time increment $h = 0.0028$ sec is selected so that $\theta = 360\ \omega h/2\pi = 19.25$ degrees. Referring to Fig. 6b it is noted that twelve steps of this increment arrive at $t = 0.0336$ sec and that the thirteenth step requires an $h' = 0.0014$ sec to complete the input of the system. The solution of the free vibrations from this point is based upon $h = 0.0028$ sec.

Since the foundation velocity is the known input, while the δ expression calls for the foundation acceleration, the latter is approximated over each short increment by the parabolic average method. That is,

$$\ddot{z}_n = \frac{\dot{z}_{n+1} - \dot{z}_n}{h}.$$

The parabolic averaging method is used for the x^3 term in the δ expression. Upon substituting the values of the parameters into Eqs. (41) for $h = 0.0028$ sec, there results

$$x_{n+1} = 0.945311\ x_n + 0.318880\ u_n - 0.054689\ \delta_n \quad (41a')$$

$$u_{n+1} = -0.318880\ x_n + 0.881534\ u_n - 0.318880\ \delta_n \quad (41b')$$

where

$$\delta_n = \frac{\dot{z}_{n+1} - \dot{z}_n}{40.320} + 0.000289\ (-x_{n-1} + 8\ x_n + 5\ x_{n+1})^3. \quad (41c')$$

The straight line averaging method is used for the first increment of the solution since x_{-1} does not exist. Table 1 shows the arrangement for solving the equations and presents the data for $n = 0$, 1, and 2. Although all numbers are carried to six decimal places, the iteration of the variable x was carried to four decimal places. This generally required four trials in each step. The results of the numerical method are plotted in comparison with the response curve obtained from an analog computer at NRL as shown in Fig. 6c.

Example 2

Figure 7a shows a two-degree-of-freedom system subjected to a transient foundation acceleration shown in Fig. 7b. There are two cubic hardening springs in the system with the following force-displacement relationships:

$$f_{10} = k_1 x_1 + \beta_1 x_1^3, \qquad f_{21} = k_2 x_2 + \beta_2 x_2^3.$$

The equations of motion are

$$\ddot{x}_1 + 2\alpha_1\omega_1\dot{x}_1 + \omega_1^2 x_1 = -\omega_1^2\delta_1 \quad (42a)$$

$$\ddot{x}_2 + 2\alpha_2\omega_2\dot{x}_2 + \omega_2^2 x_2 = -\omega_2^2\delta_2 \quad (42b)$$

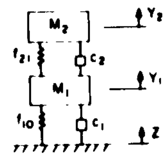
where

$$\delta_1 = \frac{\beta_1}{k_1}x_1^3 - \frac{c_2}{k_1}\dot{x}_2 - \frac{k_2}{k_1}x_2 - \frac{\beta_2}{k_1}x_2^3 + \frac{\ddot{z}}{\omega_1^2}$$

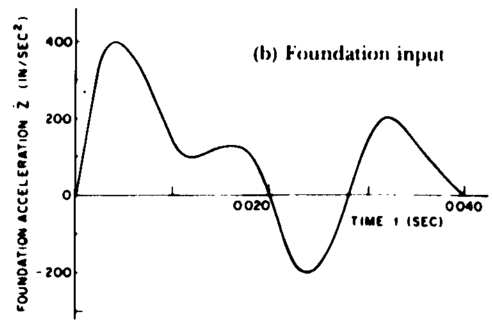
$$\delta_2 = \frac{m_2}{m_1}x_2 + \frac{\beta_2}{k_2}\left(1 + \frac{m_2}{m_1}\right)x_2^3 + \frac{c_2\dot{x}_2}{m_1\omega_2^2} - \frac{c_1\dot{x}_1}{m_1\omega_2^2} - \frac{\omega_1^2}{\omega_2^2}x_1 - \frac{\beta_1 x_1^3}{m_1\omega_2^2}.$$

TABLE 1
Arrangement for Solving Eqs. (41′)

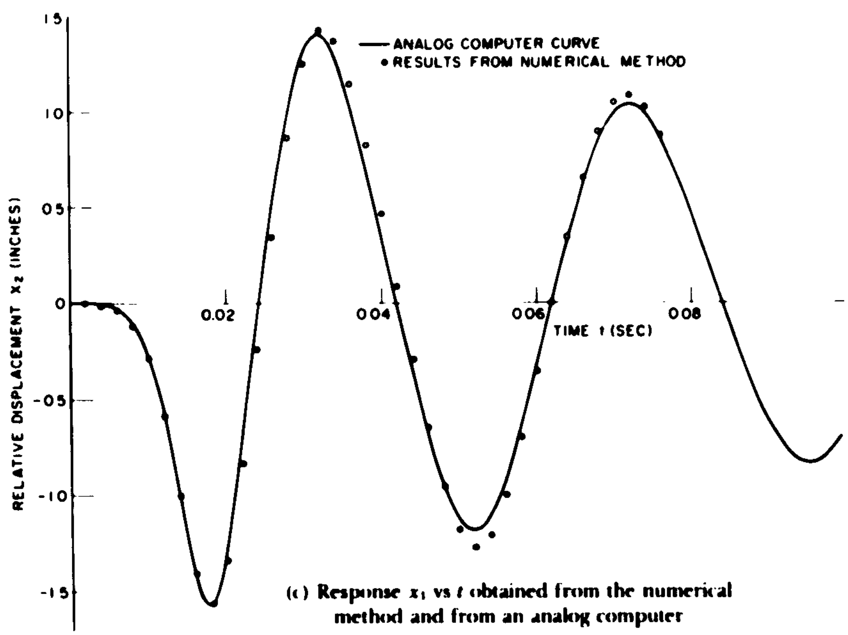
n	0.945311 x_n	0.318880 u_n	−0.054689 δ_n	x_{n+1}	−0.318880 x_r	0.881534 u_n	−0.318880 δ_n	u_{n+1}	$-x_{n-1}$	$8x_n$	$5x_{n+1}$	0.000289 $(\Sigma x)^3$	$\gamma_n \omega^2$	δ_n
0	0	0	−0.203427	−0.203427	0	0	−1.186142	−1.186142	*Straight line averaging method used, three trials required.*				3.720238	3.719712
1	−0.192302	−0.378237	−0.203427	−0.773966	0.064869	−1.045625			0	−1.627416	−3.869830	−0.048010	0.310020	0.262010
			−0.014329	−0.584868							−2.924340	−0.027254		0.282766
			−0.015464	−0.586003							−2.930015	−0.027356		0.282664
			−0.015459	−0.585998			−0.090136	−1.070892			−2.929990	−0.027356		0.282664
2	−0.553950	−0.341486	−0.015459	−0.910895	0.186863	−0.944028			0.203427	−4.687984	−4.554475	−0.213434	0.310020	0.096586
			−0.005282	−0.900718							−4.503590	−0.209850		0.100170
			−0.005478	−0.900914							−4.504570	−0.209918		0.100102
			−0.005474	−0.900910			−0.031921	−0.789086			−4.504550	−0.209917		0.100103



(a) System

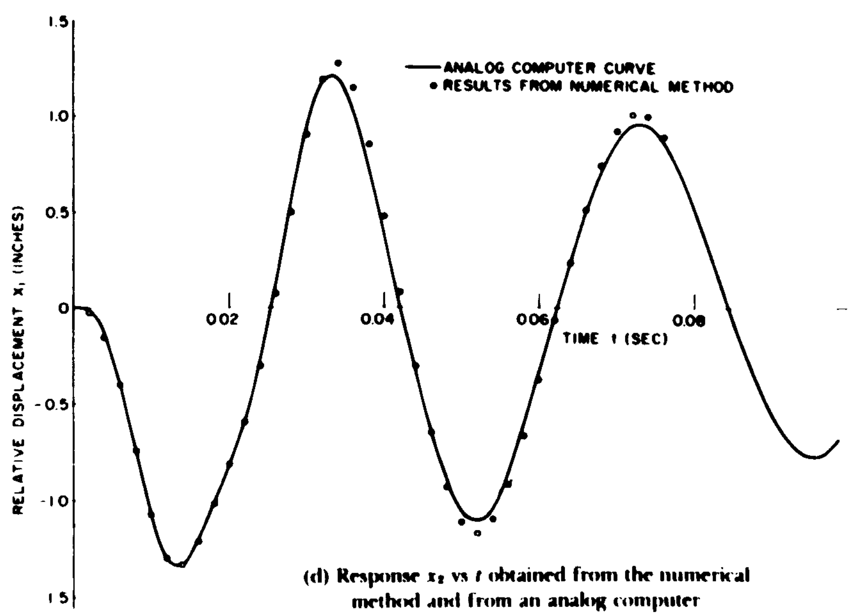


(b) Foundation input



(c) Response x_1 vs t obtained from the numerical method and from an analog computer

Fig. 7 - Input and numerical solution of a nonlinear two-degree-of-freedom system with viscous damping



(d) Response x_2 vs t obtained from the numerical method and from an analog computer

The numerical integration equations for Eqs. (42) are the same as in Example 1 with the additional requirement that the proper subscripts be used. The values of the parameters are

$m_1 = 0.40$ lb-sec²/in. $\quad m_2 = 0.10$ lb-sec²/in.

$k_1 = 8000$ lb/in. $\quad k_2 = 2500$ lb/in.

$\beta_1 = 7000$ lb/in.³ $\quad \beta_2 = 2000$ lb/in.³

$c_1 = 12$ lb-sec/in. $\quad c_2 = 3$ lb-sec/in.³

A time increment $h = 0.002$ sec is selected so that $\theta_1 = 360\ \omega_1 h/2\pi = 16.21$ degrees and $\theta_2 = 360\ \omega_2 h/2\pi = 18.12$ degrees. The parabolic averaging method is used for the variables in the δ expressions (including the input, Fig. 7b). Upon substituting the values of the parameters into the numerical integration equations, there results

$$(x_1)_{n+1} = 0.961066\ (x_1)_n + 0.270884\ (u_1)_n - 0.038934\ (\delta_1)_n$$

$$(u_1)_{n+1} = -0.270884\ (x_1)_n + 0.903603\ (u_1)_n - 0.270884\ (\delta_1)_n$$

where

$$(\delta_1)_n = 0.875000\ x_1^3 - 0.059293\ \bar{u}_2 - 0.312500\ \bar{\ddot{x}}_2 - 0.250000\ x_2^3 + \frac{\ddot{z}}{20{,}000}$$

and

$$(x_2)_{n+1} = 0.951391\ (x_2)_n + 0.301839\ (u_2)_n - 0.048609\ (\delta_2)_n$$

$$(u_2)_{n+1} = -0.301839\ (x_2)_n + 0.894121\ (u_2)_n - 0.301839\ (\delta_2)_n$$

where

$$(\delta_2)_n = 0.250000\ x_2 + x_2^3 + 0.047434\ \bar{u}_2 - 0.169706\ \bar{u}_1 - 0.800000\ x_1 - 0.700000\ x_1^3.$$

The bars above the variables in the δ expressions represent parabolic average values during the increment n to $n + 1$. The convergence of the variables of the iteration method was carried to four decimal places. This generally required five trials in each step. The numerical results are plotted against the response curves obtained from an analog computer at NRL and are shown in Figs. 7c and 7d.

CONCLUSIONS

A numerical integration method has been presented which is easily understood and provides a good solution for the transient response of nonlinear systems. Generally the smaller the increment selected for h, the greater the accuracy obtained in the response solution. Experience has shown that angular increments $\theta = 360\ \omega h/2\pi$ between 15 and 30 degrees for single-degree-of-freedom systems are generally acceptable while for two-degree-of-freedom systems θ's should range between 10 and 20 degrees. The two examples in this report used these ranges of values for θ and the reader can see the difference between the analog computer response and the numerical results.

While the examples used herein are mechanical models of structures subjected to foundation motion, the numerical method is applicable for finding the transient response of a set of nonlinear nonautonomous differential equations for other types of physical systems.

ACKNOWLEDGMENTS

The authors acknowledge R. Schreibert for his assistance with the numerical computations, and W. P. DeWitt and his staff for the analog computer solutions.

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APPENDIX A
MEAN VALUE DETERMINATION OF A FUNCTION

There are two common graphical methods for finding the mean value of the function shown in Fig. A1 for the increment $\Delta t = h$. The first method is to draw a line in an arbitrary direction by eye such that the sum of the enclosed areas between the curve and the straight line is zero. The mean value of the function is the distance from the abscissa axis to the straight line at the middle of the interval. For example, line *cd* in Fig. A1 is such a line and the point *f* on the line determines the mean value $\bar{F}$.

The second method consists of drawing the line *ab* which connects the end points of the curve segment. At the middle of the segment measure the distance between the curve and the line, that is, *eg*. Measure off 2/3 of this distance above (or below) point *g* to establish the mean value at point *f*. This second method is accurate if the curve is any part of a quadratic parabola.

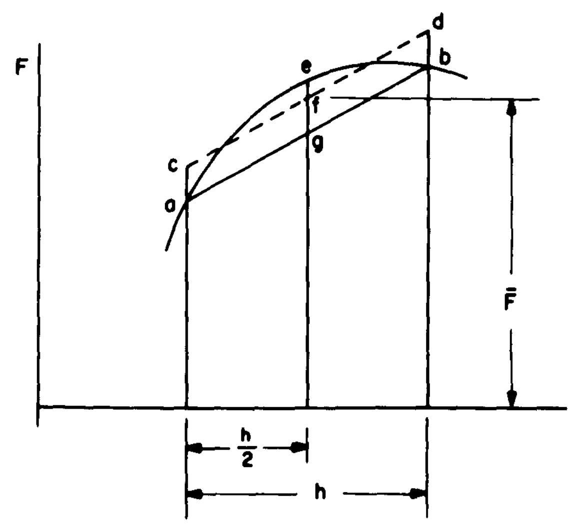

Fig.A1 - Determination of the mean value of a function $F(t)$ for the increment h

APPENDIX B
NUMERICAL INTEGRATION EQUATIONS FOR LINEARLY DAMPED SYSTEMS

Numerical integration equations are presented for the differential equation

$$\ddot{y} + 2\alpha\omega\dot{y} + \omega^2 y = \frac{F}{m} \tag{B1}$$

where $-\infty < \alpha < \infty$, $\omega > 0$. The region of α is broken into seven cases as follows:

Case I: $\alpha = 0$

Case II: $0 < \alpha < 1$

Case III: $\alpha = 1$

Case IV: $\alpha > 1$

Case V: $-1 < \alpha < 0$

Case VI: $\alpha = -1$

Case VII: $\alpha < -1$.

The applied force is represented by the parabolic method, so that

$$F = F_n + S_n \frac{t}{h} + \frac{S^2_{n-1}}{2}\left(\frac{t^2}{h^2} - \frac{t}{h}\right).$$

For foundation motion Eq. (B1) becomes

$$\ddot{x} + 2\alpha\omega\dot{x} + \omega^2 x = -\ddot{z}. \tag{B2}$$

By means of similarity between Eqs. (B1) and (B2), it is only necessary to change the following in the numerical integration equations for applied forces to obtain those for foundation motion. Let

$$y_n = x_n, \quad v_n = u_n, \quad F_n = -m\ddot{z}_n,$$

$$\frac{F_n}{k} = -\frac{\ddot{z}_n}{\omega^2}, \quad \frac{S_n}{k} = -\frac{\mathbf{S}_n}{\omega^2}, \quad \frac{S^2_{n-1}}{k} = -\frac{\mathbf{S}^2_{n-1}}{\omega^2}.$$

If the input is given by the foundation velocity, which is represented by the parabolic method, the derivative of $\dot{z}$ gives the foundation acceleration as

$$\ddot{z} = \frac{\mathbf{S}_n}{h} + \frac{\mathbf{S}_{n-1}^2}{2}\left(\frac{2t}{h} - \frac{1}{h}\right)$$

where

$$\mathbf{S}_n = \dot{z}_{n+1} - \dot{z}_n$$

$$\mathbf{S}_{n-1} = \dot{z}_{n+1} - 2\,\dot{z}_n + \dot{z}_{n-1}.$$

Equations (B1) and (B2) are solved for each of the seven cases.

CASE I: $\alpha = 0$

Input - Applied Force or Foundation Acceleration

$$y_{n+1} = y_n \cos\theta + v_n \sin\theta + \frac{F_n}{k}(1 - \cos\theta)$$

$$+ \frac{S_n}{k}\left(1 - \frac{\sin\theta}{\theta}\right) + \frac{S_{n-1}^2}{2k}\left[\frac{\sin\theta}{\theta} - \frac{2(1 - \cos\theta}{\theta^2}\right]$$

$$v_{n+1} = -y_n \sin\theta + v_n \cos\theta$$

$$+ \frac{F_n}{k}\sin\theta + \frac{S_n}{k}\left(\frac{1 - \cos\theta}{\theta}\right)$$

$$+ \frac{S_{n-1}^2}{2k}\left(\frac{1 + \cos\theta}{\theta} - \frac{2\sin\theta}{\theta^2}\right).$$

Input - Foundation Velocity

$$x_{n+1} = x_n \cos\theta + u_n \sin\theta - \frac{\mathbf{S}_n}{\omega}\left(\frac{1 - \cos\theta}{\theta}\right)$$

$$- \frac{\mathbf{S}_{n-1}^2}{\omega}\left(\frac{1 + \cos\theta}{2\theta} - \frac{\sin\theta}{\theta^2}\right)$$

$$u_{n+1} = -x_n \sin\theta + u_n \cos\theta - \frac{\mathbf{S}_n \sin\theta}{\omega\theta}$$

$$- \frac{\mathbf{S}_{n-1}^2}{\omega}\left(\frac{1 - \cos\theta}{\theta^2} - \frac{\sin\theta}{2\theta}\right).$$

CASE II: $0 < \alpha < 1$ (Let $r = \sqrt{1 - \alpha^2}$)

Input - Applied Force or Foundation Acceleration

$$y_{n+1} = y_n e^{-\alpha\theta}\left(\cos r\theta + \frac{\alpha}{r}\sin r\theta\right)$$

$$+ v_n e^{-\alpha\theta}\frac{\sin r\theta}{r}$$

$$+ \frac{F_n}{k}\left[1 - e^{-\alpha\theta}\left(\cos r\theta + \frac{\alpha}{r}\sin r\theta\right)\right]$$

$$+ \frac{S_n}{k}\left[1 - \frac{2\alpha}{\theta}(1 - e^{-\alpha\theta}\cos r\theta)\right.$$

$$\left. - (1 - 2\alpha^2)\,e^{-\alpha\theta}\frac{\sin r\theta}{r\theta}\right]$$

$$+ \frac{S_{n-1}^2}{2k}\left\{-\frac{4\alpha}{\theta} - \left[\frac{2(1 - 4\alpha^2)}{\theta^2} - \frac{2\alpha}{\theta}\right]\right.$$

$$(1 - e^{-\alpha\theta}\cos r\theta)$$

$$+ \left[\frac{1 - 2\alpha^2}{\theta} + \frac{2\alpha(3 - 4\alpha^2)}{\theta^2}\right]$$

$$\left. e^{-\alpha\theta}\frac{\sin r\theta}{r}\right\}$$

$$v_{n+1} = -y_n e^{-\alpha\theta}\frac{\sin r\theta}{r} + v_n e^{-\alpha\theta}\left(\cos r\theta - \frac{\alpha}{r}\sin r\theta\right)$$

$$+ \frac{F_n}{k}e^{-\alpha\theta}\frac{\sin r\theta}{r}$$

$$+ \frac{S_n}{k}\left(\frac{1 - e^{-\alpha\theta}\cos r\theta}{\theta} - \frac{\alpha e^{-\alpha\theta}\sin r\theta}{r\theta}\right)$$

$$+ \frac{S_{n-1}^2}{2k}\left\{\frac{2}{\theta} - \left(\frac{4\alpha}{\theta} + 1\right)\left(\frac{1 - e^{-\alpha\theta}\cos r\theta}{\theta}\right)\right.$$

$$\left. - \left[\frac{2(1 - 2\alpha^2)}{\theta^2} - \frac{\alpha}{\theta}\right]\frac{e^{-\alpha\theta}\sin r\theta}{r}\right\}.$$

Input - Foundation Velocity

$$x_{n+1} = x_n e^{-\alpha\theta}\left(\cos r\theta + \frac{\alpha}{r}\sin r\theta\right) + u_n e^{-\alpha\theta}\frac{\sin r\theta}{r}$$

$$- \frac{\mathbf{S}_n}{\omega}\left(\frac{1 - e^{-\alpha\theta}\cos r\theta}{\theta} - \frac{\alpha\, e^{-\alpha\theta}\sin r\theta}{r\theta}\right)$$

$$- \frac{\mathbf{S}_{n-1}^2}{\omega}\left\{\frac{1}{\theta} - \left(\frac{1}{2} + \frac{2\alpha}{\theta}\right)\left(\frac{1 - e^{-\alpha\theta}\cos r\theta}{\theta}\right)\right.$$

$$\left. + \left[\frac{\alpha}{2} - \frac{(1 - 2\alpha^2)}{\theta}\right]\frac{e^{-\alpha\theta}\sin r\theta}{r\theta}\right\}$$

$$u_{n+1} = -x_n e^{-\alpha\theta} \frac{\sin r\theta}{r} + u_n e^{-\alpha\theta}\left(\cos r\theta - \frac{\alpha}{r}\sin r\theta\right)$$

$$- \frac{S_n}{\omega} e^{-\alpha\theta} \frac{\sin r\theta}{r\theta} \quad - \frac{S^2_{n-1}}{\omega}\left[\frac{1 - e^{-\alpha\theta}\cos r\theta}{\theta^2}\right.$$

$$\left. -\left(\frac{\alpha}{\theta} + \frac{1}{2}\right)\frac{e^{-\alpha\theta}\sin r\theta}{r\theta}\right].$$

CASE III: $\alpha = 1$

Input - Applied Force or Foundation Acceleration

$$y_{n+1} = y_n(1+\theta)\, e^{-\theta} + v_n\theta\, e^{-\theta}$$

$$+ \frac{F_n}{k}\left[1 - (1+\theta)\, e^{-\theta}\right]$$

$$+ \frac{S_n}{k}\left[(1 + e^{-\theta}) - \frac{2}{\theta}(1 - e^{-\theta})\right]$$

$$+ \frac{S^2_{n-1}}{2k}\left[\frac{6}{\theta^2}(1 - e^{-\theta}) - \frac{2}{\theta}(1 - 2e^{-\theta}) - e^{-\theta}\right]$$

$$v_{n+1} = -y_n\theta e^{-\theta} + v_n(1-\theta)e^{-\theta} + \frac{F_n}{k}\theta e^{-\theta}$$

$$+ \frac{S_n}{k}\left(\frac{1 - e^{-\theta}}{\theta} - e^{-\theta}\right)$$

$$+ \frac{S^2_{n-1}}{2k}\left[\frac{1}{\theta} - \frac{4}{\theta^2}(1 - e^{-\theta}) + e^{-\theta}\left(1 + \frac{3}{\theta}\right)\right].$$

Input - Foundation Velocity

$$x_{n+1} = x_n(1+\theta)e^{-\theta} + u_n\theta e^{-\theta}$$

$$- \frac{S_n}{\omega}\left(\frac{1 - e^{-\theta}}{\theta} - e^{-\theta}\right)$$

$$- \frac{S^2_{n-1}}{\omega}\left[\frac{1}{\theta} - \left(\frac{1}{2} + \frac{2}{\theta}\right)\left(\frac{1 - e^{-\theta}}{\theta}\right)\right.$$

$$\left. + \left(\frac{1}{2} + \frac{1}{\theta}\right)e^{-\theta}\right]$$

$$u_{n+1} = -x_n\theta e^{-\theta} + u_n(1-\theta)e^{-\theta} - \frac{S_n}{\omega}e^{-\theta}$$

$$- \frac{S^2_{n-1}}{\omega}\left[\frac{1 - e^{-\theta}}{\theta^2} - \left(\frac{1}{2} + \frac{1}{\theta}\right)e^{-\theta}\right].$$

CASE IV: $\alpha > 1$ (Let $q = \sqrt{\alpha^2 - 1}$)

Input - Applied Force or Foundation Acceleration

$$y_{n+1} = y_n e^{-\alpha\theta}\left(\cosh q\theta + \frac{\alpha}{q}\sinh q\theta\right)$$

$$+ v_n e^{-\alpha\theta}\frac{\sinh q\theta}{q}$$

$$+ \frac{F_n}{k}\left[1 - e^{-\alpha\theta}\left(\cosh q\theta + \frac{\alpha}{q}\sinh q\theta\right)\right]$$

$$+ \frac{S_n}{k}\left[1 - \frac{2\alpha}{\theta}(1 - e^{-\alpha\theta}\cosh q\theta)\right.$$

$$\left. - (1 - 2\alpha^2)e^{-\alpha\theta}\frac{\sinh q\theta}{q\theta}\right]$$

$$+ \frac{S^2_{n-1}}{2k}\left\{-\frac{4\alpha}{\theta} - \left[\frac{2(1 - 4\alpha^2)}{\theta^2} - \frac{2\alpha}{\theta}\right]\right.$$

$$(1 - e^{-\alpha\theta}\cosh q\theta)$$

$$+ \left[\frac{1 - 2\alpha^2}{\theta} + \frac{2\alpha(3 - 4\alpha^2)}{\theta^2}\right]$$

$$\left. e^{-\alpha\theta}\frac{\sinh q\theta}{q}\right\}$$

$$v_{n+1} = -y_n e^{-\alpha\theta}\frac{\sinh q\theta}{q}$$

$$+ v_n e^{-\alpha\theta}\left(\cosh q\theta - \frac{\alpha}{q}\sinh q\theta\right)$$

$$+ \frac{F_n}{k}e^{-\alpha\theta}\frac{\sinh q\theta}{q}$$

$$+ \frac{S_n}{k}\left(\frac{1 - e^{-\alpha\theta}\cosh q\theta}{\theta} - \frac{\alpha e^{-\alpha\theta}\sinh q\theta}{q\theta}\right)$$

$$+ \frac{S^2_{n-1}}{2k}\left\{\frac{2}{\theta} - \left(1 + \frac{4\alpha}{\theta}\right)\left(\frac{1 - e^{-\alpha\theta}\cosh q\theta}{\theta}\right)\right.$$

$$- \left[\frac{2(1 - 2\alpha^2)}{\theta^2} - \frac{\alpha}{\theta}\right]$$

$$\left. e^{-\alpha\theta}\frac{\sinh q\theta}{q}\right\}.$$

Input - Foundation Velocity

$$x_{n+1} = x_n e^{-\alpha\theta}\left(\cosh q\theta + \frac{\alpha}{q}\sinh q\theta\right)$$

$$+ u_n e^{-\alpha\theta}\frac{\sinh q\theta}{q}$$

$$- \frac{S_n}{\omega}\left(\frac{1 - e^{-\alpha\theta}\cosh q\theta}{\theta} - \frac{\alpha e^{-\alpha\theta}\sinh q\theta}{q\theta}\right)$$

$$- \frac{S_{n-1}^2}{\omega}\left\{\frac{1}{\theta} - \left(\frac{1}{2} + \frac{2\alpha}{\theta}\right)\left(\frac{1 - e^{-\alpha\theta}\cosh q\theta}{\theta}\right)\right.$$

$$+ \left[\frac{\alpha}{2} - \frac{(1 - 2\alpha^2)}{\theta}\right]$$

$$\left. e^{-\alpha\theta}\frac{\sinh q\theta}{q\theta}\right\}$$

$$u_{n+1} = -x_n e^{-\alpha\theta}\frac{\sinh q\theta}{q}$$

$$+ u_n e^{-\alpha\theta}\left(\cosh q\theta - \frac{\alpha}{q}\sinh q\theta\right)$$

$$- \frac{S_n}{\omega} e^{-\alpha\theta}\frac{\sinh q\theta}{q\theta}$$

$$- \frac{S_{n-1}^2}{\omega}\left[\frac{1 - e^{-\alpha\theta}\cosh q\theta}{\theta^2}\right.$$

$$\left. - \left(\frac{\alpha}{\theta} + \frac{1}{2}\right) e^{-\alpha\theta}\frac{\sinh q\theta}{q\theta}\right].$$

CASE V: $-1 < \alpha < 0$

Use the equations for Case II but replace α by $-\alpha$.

CASE VI: $\alpha = -1$

Input - Applied Force or Foundation Acceleration

$$y_{n+1} = y_n(1 - \theta)e^{\theta} + v_n\theta e^{\theta}$$

$$+ \frac{F_n}{k}[1 - (1 - \theta)e^{\theta}]$$

CASE VI (continued)

$$+ \frac{S_n}{k}\left[1 + e^{\theta} + \frac{2}{\theta}(1 - e^{\theta})\right]$$

$$+ \frac{S_{n-1}^2}{2k}\left[\frac{6}{\theta^2}(1 - e^{\theta}) + \frac{2}{\theta}(1 + 2e^{\theta}) - e^{\theta}\right]$$

$$v_{n+1} = -y_n\theta e^{\theta} + v_n(1 + \theta)e^{\theta}$$

$$+ \frac{F_n}{k}\theta e^{\theta} + \frac{S_n}{k}\left(\frac{1 - e^{\theta}}{\theta} + e^{\theta}\right)$$

$$+ \frac{S_{n-1}^2}{2k}\left[\frac{1}{\theta} + \frac{4}{\theta^2}(1 - e^{\theta}) - e^{\theta}\left(1 - \frac{3}{\theta}\right)\right].$$

Input - Foundation Velocity

$$x_{n+1} = x_n(1 - \theta)e^{\theta} + u_n\theta e^{\theta}$$

$$- \frac{S_n}{\omega}\left(\frac{1 - e^{\theta}}{\theta} + e^{\theta}\right)$$

$$- \frac{S_{n-1}^2}{\omega}\left[\frac{1}{\theta} - \left(\frac{1}{2} - \frac{2}{\theta}\right)\left(\frac{1 - e^{\theta}}{\theta}\right)\right.$$

$$\left. - \left(\frac{1}{2} - \frac{1}{\theta}\right)e^{\theta}\right]$$

$$u_{n+1} = -x_n\theta e^{\theta} + u_n(1 + \theta)e^{\theta} - \frac{S_n}{\omega}e^{\theta}$$

$$- \frac{S_{n-1}^2}{\omega}\left[\frac{1 - e^{\theta}}{\theta^2} + \left(\frac{1}{\theta} - \frac{1}{2}\right)e^{\theta}\right].$$

CASE VII: $\alpha < -1$

Use the equations for Case IV but replace α by $-\alpha$.

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1. Nonlinear systems
2. Models – Dynamic characteristics – Mathematical analysis

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